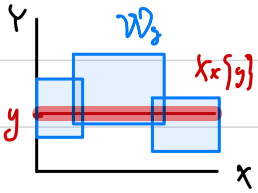
Math 451: Introduction to General Topology
Lecture 24

Thm (Finite Tychonoff). Finite products of compact spaces are compact.

Proof. By induction, it suffices to prove that if top. spaces X, Y are compact, then so is $X \times Y$.
Let $\mathcal{W}$ be an open cover of $X \times Y$. For each fixed $y \in Y$, the y-slice $X \times \{y\}$ is compact because it is homeomorphic to X via the map $x \mapsto (x,y) : X \to X \times \{y\}$. Thus, $X \times \{y\}$ admits a finite subcover $\mathcal{W}_y \subseteq \mathcal{W}$, i.e. $\mathcal{W}_y$ is finite and $X \times \{y\} \subseteq \bigcup \mathcal{W}_y = \bigcup_{W \in \mathcal{W}_y} W$. By the tube lemma, $\exists$ open $V_y \ni y$ such that $X \times V_y \subseteq \bigcup \mathcal{W}_y$. Now $\{V_y\}_{y \in Y}$ is an open cover of Y, so there is a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$. But then $X \times Y = X \times (\bigcup_{i=1}^{n} V_{y_i}) =$
$= \bigcup_{i=1}^{n} X \times V_{y_i} \subseteq \bigcup_{i=1}^{n} \bigcup \mathcal{W}_{y_i}$ so $\bigcup_{i=1}^{n} \mathcal{W}_{y_i}$ is a finite subcover of $\mathcal{W}$ for $X \times Y$. □



We now prove for arbitrary products. To do so we need an important lemma which makes it much easier to prove compactness as it gives us more control over open covers. To prove this important lemma, we need the following equivalent statement to AC:

Def. A binary relation $<$ on a set X is called a (strict) partial order if for all $x, y, z \in X$,

(i) $x \not< x$;

(ii) $x < y \Rightarrow y \not< x$;

(iii) $x < y$ and $y < z \Rightarrow x < z$.

It is called a (strict) total order if $\forall$ distinct $x, y \in X$, $x < y$ or $y < x$. A subset Y of a partially ordered set $(X, <)$ is called a chain if $<$ is a total order on Y.

Example. On $\mathcal{P}(\mathbb{N})$, the relation $\subsetneq$ is a strict partial order, which is not total because $2\mathbb{N}$ and $3\mathbb{N}$ are incomparable. The collection $\{\{0, 1, \dots, n\}\}_{n \in \mathbb{N}}$ is a chain, whose upper bound is $\mathbb{N}$.

Zorn's lemma ($\Leftrightarrow$ AC). Let $(X, <)$ be a partially ordered set. If every chain in X has an upper bound, then X has a maximal element.

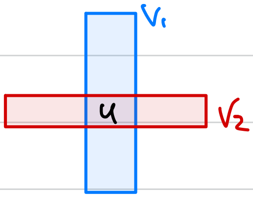
Example. Every vector space V has a basis.

Proof. Let $X :=$ the collection of all independent subsets of V ordered by $\subseteq$. Then every chain $\mathcal{C}$ in X has an upper bound, namely, $\bigcup \mathcal{C}$, which is still independent because linear dependence is witnessed by finitely many vectors, which hence be contained in a single $C \in \mathcal{C}$, a contradiction. So Zorn's lemma applies and gives a maximal independent set $B \subseteq V$. This B is a basis because for each vector $v \in V$, if $v \notin B$, then by maximality, $B \cup \{v\}$ is dependent, so v is a finite linear combination of some vectors in B. $\square$

Recall that in checking compactness it suffices to only consider open covers with sets in some basis. The following is a significant strengthening of this:

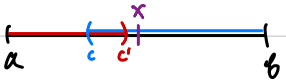
Alexander's prebasis (subbase) theorem (AC). A top space X is compact $\Leftrightarrow$ for some prebasis $\mathcal{P}$ of X, every open cover $\mathcal{U} \subseteq \mathcal{P}$ has a finite subcover.

Proof. To prove $\Leftarrow$, it suffices to show that every open cover of X with sets in the basis $\mathcal{B} := \overset{\text{(fin)}}{\bigcap} \mathcal{P} := \{V_1 \cap V_2 \cap \dots \cap V_n : \{V_1, \dots, V_n\} \subseteq \mathcal{P}, n \in \mathbb{N}\}$. Let $\mathcal{U} \subseteq \mathcal{B}$ be a cover of X and suppose towards a contradiction that there is no finite subcover. A clever application of Zorn's lemma gives an open cover $\tilde{\mathcal{U}} \subseteq \mathcal{B}$ which doesn't admit a finite subcover and is maximal such in the following sense: recall that each $U \in \tilde{\mathcal{U}}$ is a finite intersection $V_1 \cap V_2 \cap \dots \cap V_n$ of sets $V_i \in \mathcal{P}$, and the maximality of $\tilde{\mathcal{U}}$ says that if we replace U with any V_i then the resulting cover $\mathcal{U} \cup \{V_i\}$ has a finite subcover. We now show that $\tilde{\mathcal{U}}$ actually has a finite subcover, thus obtaining a contradiction. Fix any $U \in \tilde{\mathcal{U}}$ and let $U = V_1 \cap V_2 \cap \dots \cap V_n$ as above, so for each $i \in \{1, \dots, n\}$, $\tilde{\mathcal{U}} \cup \{V_i\}$ has a finite subcover $\tilde{\mathcal{U}}_i \cup \{V_i\}$, where $\tilde{\mathcal{U}}_i$ is a finite subset of $\tilde{\mathcal{U}}$. Then $X \setminus V_i$ is covered by $\tilde{\mathcal{U}}_i$ so $\bigcup_{i=1}^{n} \tilde{\mathcal{U}}_i$ covers $\bigcup_{i=1}^{n} (X \setminus V_i) = \bigcup_{i=1}^{n} V_i^c = (\bigcap_{i=1}^{n} V_i)^c = U^c$, so $\{U\} \cup \bigcup_{i=1}^{n} \tilde{\mathcal{U}}_i$ is a finite subcover of $\tilde{\mathcal{U}}$, a contradiction. $\square$



Example. We use Alexander's theorem to give an easy proof that $[a,b] \subseteq \mathbb{R}$ is compact.

Proof. Note that $\mathcal{P} := \{[a,c) : c \in (a,b)\} \cup \{(c,b] : c \in (a,b)\}$ is a prebasis for the top of $[a,b]$; indeed, any open $(c_1,c_2) = [a,c_2) \cap (c_1,b]$. To show that $[a,b]$ is compact, take a cover $\mathcal{U} \subseteq \mathcal{P}$ of $[a,b]$. Let $x := \sup\{c \in (a,b) : [a,c) \in \mathcal{U}\}$.



Then x is covered by $\mathcal{U}$ so $\exists\, U \in \mathcal{U}$ s.t. $x \in U$. This U cannot be of the form $[a,c)$ because then $c > x$ contradicting that x is the supremum. Thus $U = (c,b]$ for some $c \in (a,b)$, hence $c < x$. By the def. of sup, $\exists\, c' > c$ s.t. $[a,c') \in \mathcal{U}$. Then $\{[a,c'), (c,b]\}$ is a finite subcover of $\mathcal{U}$. □

Tychonoff's theorem ($\Leftrightarrow$ AC). An arbitrary product of compact spaces is compact.

Proof. Let $X := \prod_{i \in I} X_i$ where the X_i are compact top spaces. Recall that 1-base cylinders $[i \mapsto U_i]$ for some open $U_i \subseteq X_i$ generate the product top, so the Alexander prebasis theorem, it suffices to show that any cover $\mathcal{U}$ of X with 1-base cylinders admits a finite subcover. For each $i \in I$, let $\mathcal{U}_i := \{U_i \subseteq X_i : [i \mapsto U_i] \in \mathcal{U}\}$. We claim that $\exists\, i \in I$ such that $\mathcal{U}_i$ is a cover of X_i. Indeed, otherwise, for each $i \in I$, $X_i \setminus \bigcup \mathcal{U}_i \neq \emptyset$, so by AC $\exists\, x \in X$ s.t. $\forall i \in I$, $x(i) \in X_i \setminus \bigcup \mathcal{U}_i$. Then x is not covered by $\mathcal{U}$ since $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$. This proves the claim.

Now let $i \in I$ be such that $\mathcal{U}_i$ is a cover of X_i. By the compactness of X_i, $\exists$ finite cover $\mathcal{U}_i' \subseteq \mathcal{U}_i$. Then $\mathcal{U}' := \{[i \mapsto U] : U \in \mathcal{U}_i'\}$. Then $\mathcal{U}'$ is a finite cover of X and is a subset of $\mathcal{U}$. □